АГ Иванов 09.10.12

Скаларно произведение

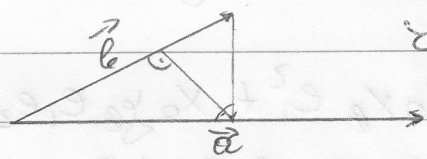
$\vec{a}\vec{b}$ - дължина $|\vec{a}||\vec{b}|\cos\sphericalangle(\vec{a}\vec{b})$ =

$|\vec{b}|\,пр_{\vec{a}}\vec{b} = |\vec{a}|\,пр_{\vec{b}}\vec{a}$

$|\vec{a}|пр_{\vec{b}}\vec{a}$ - модулът а и проекция на $\vec{a}$ върху $\vec{b}$

$\vec{a}\vec{b}=0$ тогава и само тогава, когато $\vec{a}\perp\vec{b}$

Свойства



черт 1

① $\vec{a}\vec{b}=\vec{b}\vec{a}$

② $\vec{a}(\vec{b}+\vec{c})=\vec{a}\vec{b}+\vec{a}\vec{c}$

③ $(\lambda\vec{a})\vec{b}=\lambda(\vec{a}\vec{b})$

④ $\vec{a}\vec{a}\geq 0$ или $\vec{a}^2\geq 0$ ($\vec{a}^2=0 \Leftrightarrow \vec{a}=\vec{0}$)

Нека

$\lambda<0$

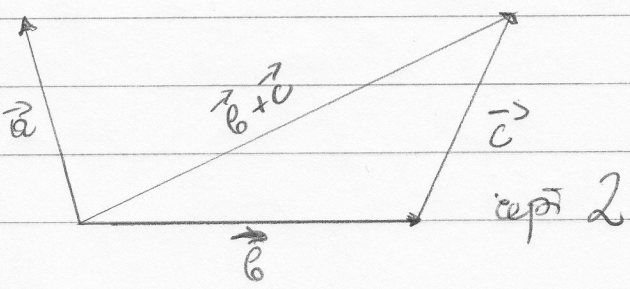
$(\lambda\vec{a})\vec{b} = |\lambda\vec{a}||\vec{b}|\cos\sphericalangle(\lambda\vec{a}\vec{b}) = |\lambda||\vec{a}||\vec{b}|\cos(\pi\sphericalangle\vec{a}\vec{b}) =$

$= -\lambda|\vec{a}||\vec{b}|\cos\sphericalangle(\vec{a}\vec{b})$ черт 1

Нека

$\vec{a}(\vec{b}+\vec{c}) = |\vec{a}|\,пр_{\vec{a}}^{\vec{b}+\vec{c}} = |\vec{a}|\left(пр_{\vec{a}}\vec{b} + пр_{\vec{a}}\vec{c}\right) \Rightarrow$

$\vec{a}\vec{b}+\vec{a}\vec{c}$ черт 2



черт 2

Ефимидово пространство - векторно пространство със скаларно произведение

Нека

$$K = e_1 e_2 e_3$$

$$\vec{a}(x_a y_a z_a) \Rightarrow \vec{a} = x_a e_1 + y_a e_2 + z_a e_3$$
$$\vec{b}(x_b y_b z_b) \Rightarrow \vec{b} = x_b e_1 + y_b e_2 + z_b e_3$$

$$\vec{a}\vec{b} = (x_a e_1 + y_a e_2 + z_a e_3)(x_b e_1 + y_b e_2 + z_b e_3) =$$

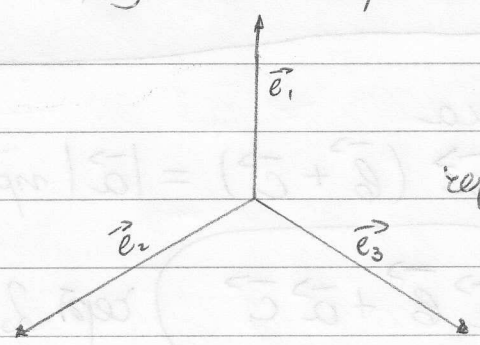
$$x_a x_b e_1^2 + x_a y_b e_1 e_2 + x_a z_b e_1 e_3 + y_a x_b e_1 e_2 + y_a y_b e_2^2 +$$
$$+ y_a z_b e_2 e_3 + z_a x_b e_1 e_3 + z_a y_b e_2 e_3 + z_a z_b e_3^2 \Rightarrow$$

ако $e_1 \perp e_2 \perp e_3 \perp e_1 - K = 0$
ортогонална

$$\boxed{\vec{a}\vec{b} = x_a x_b e_1^2 + y_a y_b e_2^2 + z_a z_b e_3^2}$$

ако $K = 0\, e_1 e_2 e_3$ - ортонормирана
(ортогонална + $|\vec{e}_1| = |\vec{e}_2| = |\vec{e}_3| = 1$) черт 3

$$\boxed{\vec{a}\vec{b} = x_a x_b + y_a y_b + z_a z_b}$$



черт 3

от $\vec{a} = x_a e_1 + y_a e_2 + z_a e_3 \quad / \cdot e_1$ $\Big\} \Rightarrow$

$(e_1 e_2 = 0 \; ; \; e_1 e_3 = 0 \; ; \; e_1^2 = 1)$

$\vec{a}\vec{e}_1 = x_a \Rightarrow |\vec{a}||\vec{e}_1| \cos \sphericalangle(a;a) = \cos\alpha$

или

$\underline{\vec{a}\vec{e}_1 = x_a = \cos\alpha}$ записваме

$$\boxed{\begin{aligned} \vec{a} &= \cos\alpha \vec{e}_1 + \cos\beta \vec{e}_2 + \cos\gamma \vec{e}_3 \\ 1 &= \vec{a}\vec{a} = \vec{a}^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma \end{aligned}}$$

$$|\vec{a}| = \sqrt{\vec{a}^2} = \sqrt{\vec{a}\vec{a}} = \sqrt{x_a^2 + y_a^2 + z_a^2}$$

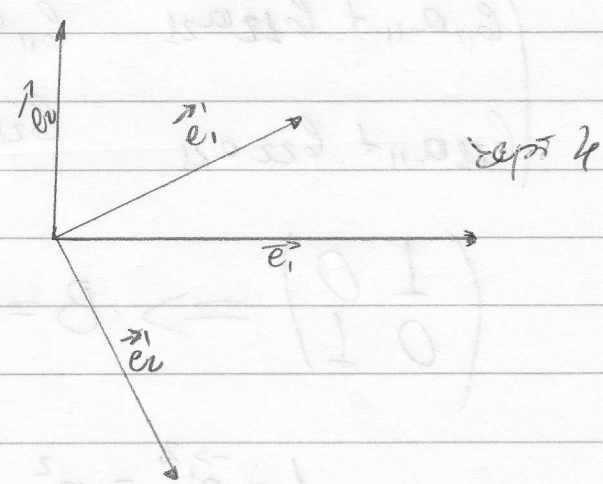
$$\cos \sphericalangle(\vec{a}\vec{b}) = \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|} = \frac{x_a x_b + y_a y_b + z_a z_b}{\sqrt{x_a^2 + y_a^2 + z_a^2}\sqrt{x_b^2 + y_b^2 + z_b^2}}$$

Смяна на координатна система в равнината

$K = O e_1 e_2$, $K' = O' = O e_1' e_2'$ черт 4

$\vec{e}_1' = a_{11}\vec{e}_1 + a_{12}\vec{e}_2$

$\vec{e}_2' = a_{21}\vec{e}_1 + a_{22}\vec{e}_2$



черт 4

т. е.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow$$

$$e' = Ae$$

$$\begin{pmatrix} \vec{e}_1' \\ \vec{e}_2' \end{pmatrix} = \begin{pmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} a_{11} e_1 + a_{12} e_2 \\ a_{21} e_1 + a_{22} e_2 \end{pmatrix}$$

$\det(A) \neq 0$ (детерминанта) защото e_1' и e_2' са линейно независими

$\vec{e}_1 = b_{11} e_1' + b_{12} e_2'$; $\vec{e}_2 = b_{12} e_1' + b_{22} e_2'$ (заместваме с A) $\Rightarrow$

$\vec{e}_1 = b_{11}(a_{11} e_1 + a_{12} e_2) + b_{12}(a_{21} \vec{e}_1 + a_{22} \vec{e}_2) =$

$= \vec{e}_1 (b_{11} a_{11} + b_{12} a_{21}) + \vec{e}_2 (b_{11} a_{12} + b_{12} a_{22})$

$\vec{e}_2 = \vec{e}_1 \underbrace{(b_{21} a_{11} + b_{22} a_{21})}_{=0} + \vec{e}_2 \underbrace{(b_{21} a_{12} + b_{22} a_{22})}_{=1}$

т.е.

$b_{11} a_{11} + b_{12} a_{21} = 1$ $\quad$ $b_{21} a_{11} + b_{22} a_{21} = 0$

$b_{11} a_{12} + b_{12} a_{22} = 0$ $\quad$ $b_{21} a_{12} + b_{22} a_{22} = 1$

$$B = \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} \Rightarrow \begin{pmatrix} b_{11} b_{12} \\ b_{21} b_{22} \end{pmatrix} \begin{pmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{pmatrix} = AB$$

$$\begin{pmatrix} b_{11} a_{11} + b_{12} a_{21} & b_{11} a_{12} + b_{12} a_{22} \\ b_{21} a_{11} + b_{22} a_{21} & b_{21} a_{12} + b_{22} a_{22} \end{pmatrix} = AB$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow B = A^{-1}$$ - обратна матрица

$1 = \vec{e}_1^{\,2} = a_{11}^2 + a_{12}^2$

$0 = e_1' e_2' = a_{11} a_{21} + a_{12} a_{22}$

$1 = \vec{e}_2^{\,2} = a_{21}^2 + a_{22}^2$

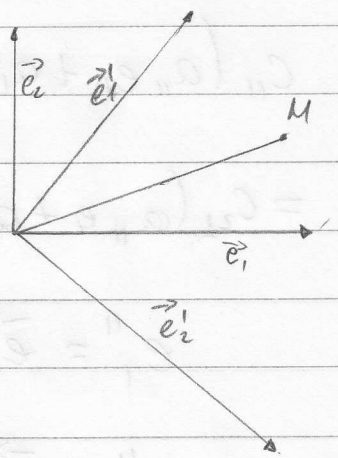
$$A^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$A^tA = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{12}a_{11} + a_{21}a_{22} & a_{11}^2 + a_{22}^2 \end{pmatrix} \Rightarrow$$

$$A^tA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ или } A^{-1} = A^t \text{ - ортогонални матрици}$$

ФАКТ $\boxed{\det(AB) = \det(A)\det(B)}$

Точка $M(x,y)$ черт 5



$\overrightarrow{OM} = x\vec{e}_1 + y\vec{e}_2 = x'\vec{e}_1' + y'\vec{e}_2'$

заместване от $\vec{e}_1'$ и $\vec{e}_2'$

$$x'(a_{11}\vec{e}_1 + a_{12}\vec{e}_2) + y'(a_{21}\vec{e}_1 + a_{22}\vec{e}_2) \Rightarrow$$

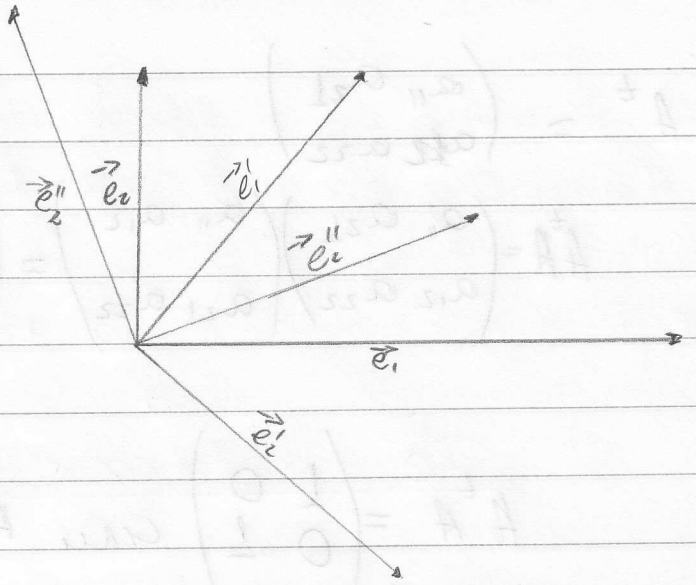
$$\vec{e}_1(x'a_{11} + a_{21}y') + \vec{e}_2(x'a_{12} + a_{22}y') \Rightarrow$$

$$\left.\begin{aligned} x &= a_{11}x' + a_{21}y' \\ y &= x'a_{12} + a_{22}y' \end{aligned}\right| = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = A^t$$

$e' = Ae \Rightarrow x = A^t x'$

$e' = Ae \mid :(-1) \Rightarrow e = A^{-1}e' \Rightarrow x' = (A^t)^{-1}x$

$K'' = O'' \equiv O e_1'' e_2'' e_3''$



$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\begin{pmatrix} e_1'' \\ e_2'' \end{pmatrix} = C \begin{pmatrix} e_1' \\ e_2' \end{pmatrix}$$

$$\begin{pmatrix} e_1'' \\ e_2'' \end{pmatrix} = \begin{pmatrix} c_{11} e_1' + c_{12} e_2' \\ c_{21} e_1' + c_{22} e_2' \end{pmatrix} = CA \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$c_{11}(a_{11} e_1 + a_{12} e_2) + c_{12}(a_{21} e_1 + a_{22} e_2) =$$

$$= c_{21}(a_{11} e_1 + a_{12} e_2) + c_{22}(a_{21} e_1 + a_{22} e_2) \Longrightarrow$$

$$e_1'' = \vec{e}_1 (c_{11} a_{11} + c_{12} a_{21}) + \vec{e}_2 (c_{11} a_{12} + c_{12} a_{22})$$

$$e_2'' = \vec{e}_1 (c_{21} a_{11} + c_{22} a_{21}) + \vec{e}_2 (c_{21} a_{12} + c_{22} a_{22})$$

Матрица на прехода

Ориентация на базис на въртене

Дефиниция

K, K' еднакво ориентирани в пространството ако $\det(A) > 0$

обратно ориентирани ако $\det(A) < 0$

Векторно произведение на 2 вектора

$$\vec{a} \times \vec{b} = \vec{c}$$

Свойства

1. $|\vec{c}| = |\vec{a}||\vec{b}| \times \sin \sphericalangle(ab)$
2. $\vec{ac} \perp \vec{a}$; $\vec{a} \times \vec{b} \perp \vec{b}$
3. $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ – положително ориентирани в $\mathbb{R}$
4. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}, \vec{b}$ – колинеарни или $\vec{a} = \lambda \vec{b}$

$(\mathbb{R}^3, \text{ор})$ $\mathbb{R}^3$ с ориентация

Смесено произведение на 3 вектора

$$(\vec{a} \times \vec{b})\vec{c}$$

Твърдение

$|(\vec{a} \times \vec{b})\vec{c}| = V_{(abc)}$ – обема на паралелепипед

$(\vec{a} \times \vec{b})\vec{c} = |\vec{a} \times \vec{b}| \, \text{пр}_{\vec{a} \times \vec{b}} \vec{c} = V$ паралелепипед

$|(\vec{a} \times \vec{c})\vec{b}| = |(\vec{a} \times \vec{b})\vec{c}|$ вместо $(\vec{a} \times \vec{b})\vec{c} = \vec{abc}$

Свойства на векторно произведение

1. $abc = bca = cab = -bac = -acb = -cba$
2. $(\lambda \vec{a}) \times \vec{b} = \lambda \vec{a} \times \vec{b}$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{ab} + \vec{ac}$

Ако $\vec{V}$ – произволен вектор то

$\vec{a} \times (\vec{b} + \vec{c})\vec{V} = -(\vec{V} \times \vec{a})(\vec{b} + \vec{c}) = \vec{a} \times \vec{b}\vec{V} + \vec{a} \times \vec{c}\vec{V} \Rightarrow$

$$(\vec{a}\times\vec{b}+\vec{a}\times\vec{c})\vec{V}$$

$$\vec{a}\times(\vec{b}+\vec{c})\vec{V}=(\vec{a}\times\vec{b}+\vec{a}\times\vec{c})\vec{V}$$

$$[\vec{a}\times(\vec{b}+\vec{c})-(\vec{a}\times\vec{b}+\vec{a}\times\vec{c})]\vec{V}=0$$

за всеки произволен вектор $\vec{V} \Rightarrow \underline{\underline{\vec{V}=0}}$

при паралелепипед

$\vec{a}\vec{b}\vec{c}=0 \Leftrightarrow V=0$ т.е. abc - компланарни